Розділ 1

АНАЛІТИЧНІ ТА ЧИСЛОВІ МЕТОДИ В МЕХАНІЦІ ТА ФІЗИЦІ РУЙНУВАННЯ БУДІВЕЛЬНИХ МАТЕРІАЛІВ І КОНСТРУКЦІЙ

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MODELING OF THE STRESS STATE IN THIN ISOTROPIC PLATES

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A method of calculating the stress state in thin isotropic rectangular arbitrarily loaded plates for various boundary conditions has been developed.

Solution of the problem is reduced to solution of a differential equation of the fourth order in particular derivatives

$$\nabla^2 \, \nabla^2 \, w = \frac{q}{D} \ , \qquad (1)$$

where ∇^2 is Laplace's differential operator; *w* – deflection of the plate; *q* – transverse load applied to the upper surface of the plate and *D* is bending rigidity of the plate.

Solution of the equation (1) is presented in the form of a sum of its particular solution w_* and the general solution w_o of corresponding uniform equation

$$w = w_o + w_* \qquad (2)$$

General solution w_o is presented as a sum of products of unknown coefficients R_{kpsv} and shape functions $W_{kpsv} \ \ x_1, x_2$.

$$w_o \ \ x_1, x_2 \ = W_{kpsv} \ \ x_1, x_2 \ \cdot R_{kpsv} \qquad (3)$$

Coefficients R_{kpsv} are treated as degrees of freedom of the plate.

Similarly, a particular solution is given as a sum of products of force functions $W_* \ \ x_1, x_2$ and other unknown coefficients.

Such approach allows to satisfy conditions at the edge and on the surface of the plate.

Boundary conditions are performed in the separate nodes at the plate edge (in each node two boundary conditions are written). In each node on the plate surface only one condition is written.

A program which automatically generates and places nodes at the edge and on the surface of the plate has been developed.

The expression (3) is called a function of the plate deflection state. State functions of other static and kinematic quantities are obtained from formula (3) by Automatic Differentiation.

Distributions of deflection, tangent displacements, moments and shearing forces are obtained on the whole area of the plate.

In this paper, two variants of the plate are considered: symmetrical and nonsymmetrical one. A plate is considered symmetrical if it satisfies the conditions of symmetry of the plate contour and boundary conditions also the conditions of symmetry of the external load and mechanical properties. Although if one of them is not performed the plate is nonsymmetrical.

For a symmetrical plate, the results are presented in the form of plots built in the central and edge cross sections of the plate. The results are compared with numerical ones obtained with the help of Package ABEQUS in the form of space plots.

It is shown that kinematic boundary conditions are performed exactly with analytical and numerical approaches. Instant results are not coincided as statistical boundary ones.

For nonsymmetrical plates, the results are given in the form of contour plots on the whole area of the plate.

Key words: mathematical model, thin isotropic plates, automatic differentiation.

Росінські К., Делявський М., Фамуляк Ю. Моделювання напруженого стану в тонких ізотропних плитах

Розроблено метод розрахунку напруженого стану в тонких ізотропних прямокутних довільно завантажених плитах для різних крайових умов.

Розв'язок задачі зведено до розв'язку диференціального рівняння четвертого порядку в частинних похідних:

$$\nabla^2\,\nabla^2\,w = \frac{q}{D}, \tag{1}$$

де ∇^2 – диференціальний оператор Лапласа; w – прогин плити; q – поперечне навантаження, прикладене до верхньої сторони плити; D – жорсткість плити на згин.

Розв'язок рівняння (1) представлено у вигляді суми його часткового розв'язку w_{*} і загального розв'язку w_o відповідного йому однорідного рівняння:

$$w = w_o + w_* \ . \tag{2}$$

Загальний розв'язок представлено у вигляді суми добутків координатних функцій $W_{kpsv}\ \ x_1, x_2$ на невідомі параметри R_{kpsv}, потрактовані як ступені свободи плити:

$$w_o\ \ x_1, x_2\ = W_{kpsv}\ \ x_1, x_2\ \cdot R_{kpsv}\ . \tag{3}$$

Подібно частковий розв'язок подано як суму добутків силових функцій $W_*\ \ x_1, x_2$ на інші невідомі коефіцієнти.

Такий підхід дозволяє виконувати умови на краю і на поверхні плити.

Крайові умови задовольняються в окремих вузлах на краю плити (в кожному вузлі записано дві умови). В кожному вузлі на поверхні плити записується одна умова.

Розроблена програма, яка автоматично утворює і розміщує вузли на краю і на поверхні плити.

Вираз (3) названо функцією стану прогину плити. Функції стану інших статичних і кінематичних величин отримано з формули (3) за допомогою автоматичного диференціювання.

Отримано розклади прогину, дотичних переміщень, моментів і поперечних сил в об'ємі плити.

Розглянуто два варіанти плити: симетричну і несиметричну. Плита вважається симетричною, якщо вона задовольняє умови симетричності контуру плити і крайових умов, а також умови симетричності зовнішнього навантаження і механічних властивостей. Якщо хоча б одна з цих умов не виконується – плита є несиметричною.

Для симетричної плити результати представлено у вигляді графіків, побудованих у центральних і крайових перерізах плити. Результати порівняно з результатами, отриманими за допомогою пакету ABAQUS у вигляді просторових графіків.

Установлено, що для кінематичних крайових умов аналітичні результати, отримані запропонованим методом і методом скінченних елементів (ABAQUS), практично збігаються, натомість спостерігається суттєва розбіжність результатів для статичних крайових умов.

Для несиметричних плит результати представлені у вигляді контурних графіків по всьому об'єму плити.

Ключові слова: математична модель, тонкі ізотропні пластини, автоматична диференціація.

Introduction. Modern development of computer technology demands construction of new design models of plate structure. These models can be divided into two groups: analytical and numerical ones. Analytical models are described by partial differential equations. As a result, solution of the structure within the analytical model caused development of various analytical methods.

The analytical methods [5; 13–15; 19] are more exact as compared to numerical ones but their possibilities are constrained to plate structure and limited by simple contours.

The base of numerical methods is functional of proper potential energy. Numerical methods [10–12; 16; 20] are less exact but they allow to solve the structure of arbitrary configuration where the analytical methods are impossible but at the expense of a significant increasing the number of unknowns. It increases the time for calculations and leads to accumulation of computational errors.

Many methods which are free from disadvantages of standard numerical ones have been suggested in recent years. They create a separate group called analytical-numerical methods [7; 8; 17]. According to these methods, a part of equations is performed analytically and others with help of numerical procedures. The separate group takes meshless methods [4; 9; 18]. The method suggested in present paper belongs to this group.

Materials and Methods. Let us consider thin isotropic rectangular plate (Fig. 1) which has thickness h and plane sizes $2a_s$, s = 1, 2. Plate is referred to right-handed Cartesian coordinate system Ox_1x_2 originated at its geometrical center and constrained by contour C.

Outer load of intensity (x_1, x_2) is applied to upper surface of the plate while its bottom surface is unloaded.

The base of analysis of the thin plate structure is made by the Kirchhoff theory. It is the approximate theory of zero order of deformation. But in many cases, the Kirchhoff theory gives satisfactory results with sufficient accuracy for the practical purpose and correctly describes behavior of the structure.

Solution of thin isotropic plates within the Kirchhoff theory is equivalent to solution of non-uniform fourth order partial differential equation

$$\nabla^2 \nabla^2 w = \frac{q}{D}, \qquad (1)$$

for given boundary conditions. Equation (1) is an equilibrium equation written in term of displacement. In the above function (x_1, x_2), there is the deflection of the plate and D its flexural rigidity. The function (x_1, x_2) describes distribution of the outer active load on upper surface of the plate.

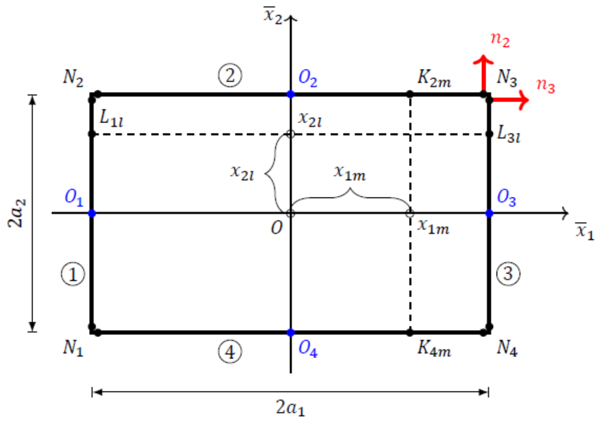


Fig. 1. Rectangular plate

Solution of the equation (1) is presented as a sum

$$w = w_o + w_* \qquad (2)$$

of general solution $w0$ of the Equation (3)

$$\nabla^2 \nabla^2 w = 0 \qquad (3)$$

and some particular solution of the Equation (1).

To solve the Equation (3) we take the form

$$w_o = f_{kps} \; x_s \; \cdot T_{kp\;3-s} \; (x_{3-s}) \qquad (4)$$

where $f_{kps}\,(x_s)$ are unknown functions; $T_{kps}\,(x_s)$ are given trigonometric ones:

$$T_{kps} \; x_1, x_2 \; = \begin{matrix} \cos \; k_{kps}\, x_s \; , p = 1, 2, \\ \sin\,(k_{kps} \; x_s), p = 3, 4, \end{matrix} \qquad (5)$$

and

$$k_{kps} = \begin{matrix} \gamma_{ks}, p = 1, 3, \\ \delta_{ks}, p = 2, 4. \end{matrix} \qquad (6)$$

Parameters γ_{ks}, δ_{ks} are calculated as

$$\gamma_{ks} = \frac{k\pi}{a_s}; \; \delta_{ks} = \frac{2k-1 \; \pi}{2a_s}. \qquad (7)$$

In the above $p = 1, \dots, 4$, $s = 1, 2$, k is number of single approximation, where $k = 1, \dots, K$ and K is several approximations of the solution and determines its accuracy. The K is greater the accuracy of the solution is better.

The Einstein summation rule is used here. According to this convention indices repeated twice in a single term imply the summation is to be done. Summation indices in formula (4) are k, p and s.

The functions $f_{kps}\,(x_s)$ including in formula (4) are taken in the following form

$$f_{kps} \; x_s \; = R_{kps} E_{kps}\,(x_s), \qquad (8)$$

where R_{kps} are unknown coefficients. Because indices k, p, s appear at the both sides of this formula, summation does not perform according to these indices.

We except the functions $E_{kps}\,(x_s)$ as

$$E_{kps} \; x_s \; = \exp\,(\lambda_{kps} x_s), \qquad (9)$$

where λ_{kps} are unknown parameters.

Substituting expressions (4), (8), (9) into equation (3) we come to the system of characteristic equations of the fourth order with respect to the parameters λ_{kps}

$$\lambda_{kps}^4 - 2k_{kp\;3-s}^2 \; \lambda_{kps}^2 + k_{kp\;3-s}^4 \; = 0. \qquad (10)$$

The parameters k_{kps} included in the equation (10) are determined with help of parameters of trigonometric functions (6).

Let's λ_{kpsv}, $v = 1, \dots, 4$ be roots of equations (10).

Finally, the functions (8) takes the form [4; 5]

$$f_{kps} \; x_s \; = R_{kpsv} \cdot B_{kpsv}\,(x_s). \qquad (11)$$

In the above functions

$$B_{kpsv} = \begin{matrix} \cos h \;\; k_{kp\;3-s} \; x_s \; , & v{=}1 \\ \frac{x_s}{a_s} \sin h \;\; k_{kp\;3-s} \; x_s \; , & v{=}2 \\ \sin h \;\; k_{kp\;3-s} \; x_s \; , & v{=}3 \\ \frac{x_s}{a_s} \cos h \;\; k_{kp\;3-s} \; x_s \; , & v{=}4 \end{matrix} \qquad (12)$$

are called the basic functions of the solution. Using the obtained relations, we present solution of the equation (3) in following compact form

$$w_o \;\; x_1, x_2 \;\; = W_{kpsv} \;\; x_1, x_2 \; \cdot R_{kpsv} \qquad (13)$$

In particular case $p = 1, 2$, $s = 1, 2$, $v = 1, 2$ we obtain symmetric model of the plate presented in the paper [5]. The introduced functions

$$W_{kpsv} \;\; x_1, x_2 \;\; =$$

$$= B_{kpsv} \;\; x_s \; \cdot T_{kp\;3-s} \; (x_{3-s}) \qquad (14)$$

are dependent on the solution of the problem and they are called shape functions of the plate deflection. The unknown coefficients R_{kpsv} are determined from boundary conditions at the plate contour.

The next substituting formula (13) into expression (2) we obtain general solution of non-uniform equation (1)

$$w \;\; x_1, x_2 \;\; = R_{kpsv} \cdot W_{kpsv} \;\; x_1, x_2 \;\; +$$

$$+ W_*\,(x_1, x_2), \qquad (15)$$

where $W_* \;\; x_1, x_2 \;\; = w_*\,(x_1, x_2)$ is its particular solution. It can be presented as

$$W_* \;\; x_1, x_2 \;\; = C_{mnpq} \cdot W_{mnpq}\,(x_1, x_2). \qquad (16)$$

The functions $W_{mnpq}\,(x_1, x_2)$ are dependent on outer load and called force functions of the plate deflection. For each specific problem they can be obtained using special procedures. In the present paper they have been taken in the form

$$W_{mnpq} \;\; x_1, x_2 \;\; = T_{mp1} \;\; x_1 \; \cdot T_{nq2} \;\; x_2 \; ,$$

$$= p, q = 1, \ldots, 4. \qquad (17)$$

The unknown parameters C_{mnpq} must be determined separately. Expressions for tangent displacements $u_1(x_1, x_2)$, $u_2(x_1, x_2)$, moments $M_{11}(x_1, x_2)$, $M_{22}(x_1, x_2)$, $M_{12}(x_1, x_2)$, shearing forces $Q_1(x_1, x_2)$, $Q_2(x_1, x_2)$ and generalized shearing forces $V_1(x_1, x_2)$, $V_2(x_1, x_2)$ were obtained by differentiating the expressions (14), (17).

In the author's computer program, the automatic differentiation (AD) is used [1–3].

The obtained expressions are presented in general form

$$F\ \ x_1, x_2\ = R_{kpsv} \cdot F_{kpsv}\ \ x_1, x_2\ + \\ + F_*\ (x_1, x_2), \qquad (18)$$

where function $F\ (x_1, x_2)$ is vector of functions $F = (U, V, X, Y, Z, \ldots)$. Next designations

$$\begin{array}{lllll} w \to W, & \varphi_1 \to U, & \varphi_2 \to V, & M_{11} \to X, & M_{22} \to Y, \\ M_{12} \to Z, & Q_1 \to G, & Q_2 \to H, & V_1 \to K, & V_2 \to L. \end{array} \qquad (19)$$

are introduced here. It is seen that expressions (18) have the same structure. Functions U_{kpsv}, V_{kpsv} and so on are partial derivatives of shape functions W_{kpsv}. They are called functions of shape of displacements, moments, shearing forces and generalized shearing forces.

Similarly, the functions U_*, V_*, etc. are partial derivatives of force functions W_* and they are called force functions of tangential displacements, moments and shearing forces.

The introduced expression (15), (18) are called functions of the state of displacements and stresses. Their set creates the design model of a plate.

Since the expression (15) is solution of the equation (1) of the internal forces in the plate be balanced with outer load applied to its upper surface. But the expressions (15), (18) are undetermined because they are dependent on arbitrary parameters R_{kpsv} which can be treated as degrees of freedom of the plate. Their number for given approximation K is

$$i = K\ psv. \qquad (20)$$

In order to stable of the plate the links in the form of boundary conditions must be imposed at its contour. The presented structure (15), (18) of solution allows easily to simulate various boundary conditions written in the separate points at the plate contour.

The total number of such conditions must be equal to number of parameters R_{kpsv} (20). Each condition corresponds to one parameter. Because $p = 4$, $s = 2$, $v = 4$ we can write $32K$ boundary conditions at the plate contour. Since at each points two boundary conditions are written there must be put $16K$ nodes at the plate contour. Unknown coefficients R_{kpsv} are determined satisfying these boundary conditions at the considered nodes.

Generation of current nodes. The consider rectangular plate in the Figure (1) has eight stable points at the contour which are called stationary nodes. They are midpoints O_i and corner points N_i, $i = 1, \ldots, 4$.

We introduce two sets $X_1 = \ x_{1m}\ $ and $X_2 = \ x_{2n}\ $ of points $\ x_{1m}, x_{2n}\ $ with numbers (m, n) uniformly placed at the axes of the plate within the intervals $X_j \in\ -a_j, 0\ \cup\ 0, a_j\ $, $j = 1, 2$ and call them initial points. Points $inf\ x_{1m}\ = x_{11} = -a_1, sup\ x_{1m}\ = x_{1\mu} = a_1$ and $inf\ x_{2n}\ = x_{21} = -a_2, sup\ x_{2n}\ = x_{2\mu} = a_2$ are limiting points.

Next, the author projects the initial points onto contour C. Each point generates only one node at the separate edge of the plate. It is evident that each edge will contain nodes generated only by either set X_1 or set X_2. From this reason it is not possible that two initial point of set X_1 and X_2 fall into one node at the edge. It means that there are unequivocal functional dependence between set of initial points and their projections onto contour of the plate.

The obtained projections are called current nodes. Nodes generated by set of points X_1 onto horizontal edges we designate K_{rm}, $r = 2,4$ and nodes placed on vertical edges as L_{rn}, $r = 1,3$. Here r is the number of the edge and m, n are numbers of initial points in corresponding sets.

Next we go to analysis of the corners points in the plate. Since all corners are the same, it is sufficient to do analysis one of them, say corner N_3. We consider limit when element x_{1m} of the set $X_1 = \ x_{1m}\ $ tends to limiting point $x_{1m} = a_1$. If $x_{1m} \to a_1$ sequence of current nodes K_{1m} approaches to corner N_3. Such limit exists always because according to the assumption limiting point a_1 exists. We designate it as K_{23}. Here first index means number of the edge and second index corresponds to corner number. We have

$$\lim_{x_{1m} \to a_1} K_{2m} = K_{23}. \qquad (21)$$

Node K_{23} is called limiting node at Edge 2 in corner N_3. Similarly, we define node at Edge 3.

$$\lim_{x_{2l} \to a_2} L_{3l} = L_{33}. \qquad (22)$$

Union of these nodes is called a simple corner node and designated it

$$S_3 = K_{23} \cup L_{33}. \qquad (23)$$

Although the obtained limiting nodes have the same coordinates $K_{23}\ \ a_1, a_2\ $, $L_{23}\ \ a_1, a_2\ $ they are different ones $K_{23} \neq L_{33}$ because lie at different edges of the normals **n**2 and **n**3. A rectangular plate has four simple corner nodes. Union of current and limiting nodes we called edge nodes. At each edge node two boundary conditions are written.

Let us note that described procedure of generation of edge nodes and their distribution at the plate contour is fulfilled with help of Author's program.

Now we calculate the number of initial points assuring given K accuracy of solution. According to (20) we have $16K$ edge nodes in approximation K uniformly distributed at the contour including limiting nodes. Half of them ($8K$) must be placed at horizontal edges and others at vertical ones. Since each initial point generates two current nodes at the opposite edges thus $4K$ points must be introduce at the axes Ox_1 and $4K$ points at the axes Ox_2. This in K approximation $8K$ initial points must be chosen.

In order to obtain a particular solution $w_*(x_1,x_2) = W_*(x_1,x_2)$ we present outer load $q(x_1,x_2)$ in the form of double trigonometric series

$$q(x_1,x_2) = q_{mnpq} \cdot T_{mnpq}(x_1,x_2), \qquad (24)$$

where $T_{mnpq}(x_1,x_2)$ are double trigonometric functions of the form

$$T_{mnpq}(x_1,x_2) = T_{mp1}(x_1) \cdot T_{nq2}(x_2),$$
$$m,n = 1,2,3,\ldots;\ p,q = 1,\ldots,4. \qquad (25)$$

For rectangular plates parameters q_{mnpq} can be determined as coefficients of expansion of the outer load in Fourier series on the surface of the plate.

Correspondingly to this force functions W_{mnpq} are taken in the same form

$$W_{mnpq}(x_1,x_2) = T_{mp1}(x_1) \cdot T_{nq2}(x_2),$$
$$p,q = 1,\ldots,4. \qquad (26)$$

Next we substitute these expressions into Equation (1). Taking into account relations (24) and equating coefficients at the same expressions in the both sides of Equation (1) we find coefficients C_{mnpq} (16) expressed over outer load (24).

Solution of the problem. The loaded functions were taken as particular case of general function (24) for $m = n = 1$; $p = q = 2$. We have

$$q(x_1,x_2) = q_{1122} \cdot T_{1122}(x_1,x_2). \qquad (27)$$

Correspondingly, a particular solution of equation (1) takes the form

$$w_*(x_1,x_2) = W_*(x_1,x_2) =$$
$$= C_{1122} \cdot W_{1122}(x_1,x_2). \qquad (28)$$

Let us introduce the designations:

$$q_{1122} = q_0,\ C_{1122} = C \qquad (29)$$

and obtain

$$q(x_1,x_2) = q_0 \cdot T_{1122}(x_1,x_2), \qquad (30)$$
$$w_*(x_1,x_2) = W_*(x_1,x_2) =$$
$$= C \cdot W_{1122}(x_1,x_2). \qquad (31)$$

Next we substitute expressions (30), (31) into equation (1). After differentiation and gathering similar expressions we find constant C.

$$C = \frac{q_0}{D\,(\delta_{11}^2 + \delta_{12}^2)^2}. \qquad (32)$$

Here q_0 is intensity of outer load and D is bending rigidity of the plate. Finally, a particular solution of equation (1) takes the form

$$w_*(x_1,x_2) = C \cdot \cos(\delta_{11} x_1) \cdot \cos(\delta_{12} x_2) =$$
$$= C \cdot \cos\left(\frac{\pi}{2a_1} x_1\right) \cdot \cos\left(\frac{\pi}{2a_2} x_2\right) \qquad (33)$$

The problem has been solved in third approximation ($K = 3$) using 12 initial points. Consequently, we have 24 edge nodes. The performed calculations have shown that increasing the K value for no longer affects the accuracy of the solution. It means that an exact solution is already obtained for small K. This confirms the high effectiveness of this method.

The results were obtained using a Python-based computer program by implementing the considered mathematical model [6].

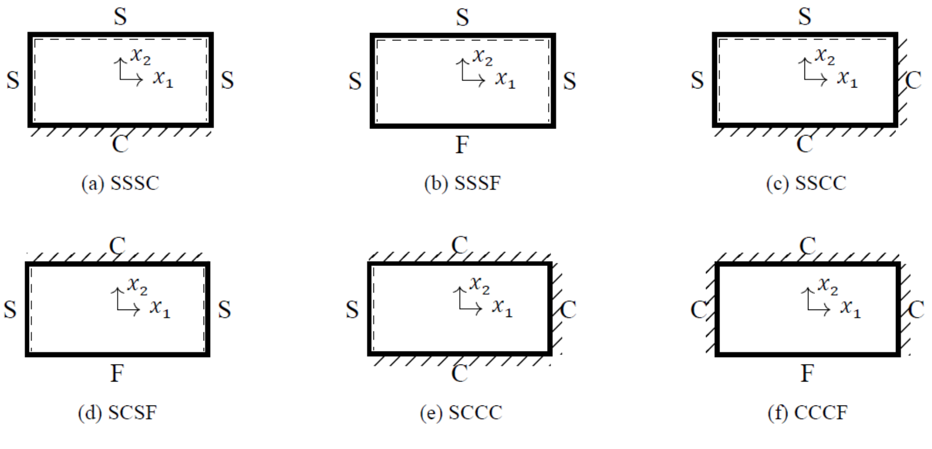


Fig. 2. Kinds of links of edges of plates

Table

Cases of boundary conditions of considered plates

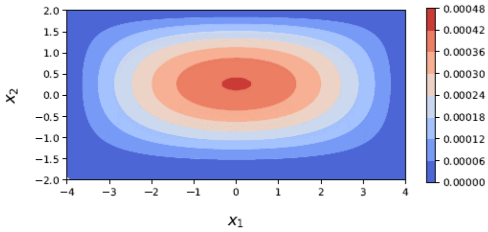
Case	Boundary conditions	Function	Case	Boundary conditions	Function
SSSC	$w(x_1, x_2)_{\,x_1=-a_1} = 0$	W	SCSF	$w(x_1, x_2)_{\,x_1=-a_1} = 0$	W
	$M_{11}(x_1, x_2)_{\,x_1=-a_1} = 0$	X		$M_{11}(x_1, x_2)_{\,x_1=-a_1} = 0$	X
	$w(x_1, x_2)_{\,x_2=a_2} = 0$	W		$w(x_1, x_2)_{\,x_2=a_2} = 0$	W
	$M_{22}(x_1, x_2)_{\,x_2=a_2} = 0$	Y		$\varphi_2(x_1, x_2)_{\,x_2=a_2} = 0$	V
	$w(x_1, x_2)_{\,x_1=a_1} = 0$	W		$w(x_1, x_2)_{\,x_1=a_1} = 0$	W
	$M_{11}(x_1, x_2)_{\,x_1=a_1} = 0$	X		$M_{11}(x_1, x_2)_{\,x_1=a_1} = 0$	X
	$w(x_1, x_2)_{\,x_2=-a_2} = 0$	W		$M_{22}(x_1, x_2)_{\,x_2=-a_2} = 0$	Y
	$\varphi_2(x_1, x_2)_{\,x_2=-a_2} = 0$	V		$V_2(x_1, x_2)_{\,x_2=-a_2} = 0$	L
SSSF	$w(x_1, x_2)_{\,x_1=-a_1} = 0$	W	SCCC	$w(x_1, x_2)_{\,x_1=-a_1} = 0$	W
	$M_{11}(x_1, x_2)_{\,x_1=-a_1} = 0$	X		$M_{11}(x_1, x_2)_{\,x_1=-a_1} = 0$	X
	$w(x_1, x_2)_{\,x_2=a_2} = 0$	W		$w(x_1, x_2)_{\,x_2=a_2} = 0$	W
	$M_{22}(x_1, x_2)_{\,x_2=a_2} = 0$	Y		$\varphi_2(x_1, x_2)_{\,x_2=a_2} = 0$	V
	$w(x_1, x_2)_{\,x_1=a_1} = 0$	W		$w(x_1, x_2)_{\,x_1=a_1} = 0$	W
	$M_{11}(x_1, x_2)_{\,x_1=a_1} = 0$	X		$\varphi_1(x_1, x_2)_{\,x_1=a_1} = 0$	U
	$M_{22}(x_1, x_2)_{\,x_2=-a_2} = 0$	Y		$w(x_1, x_2)_{\,x_2=-a_2} = 0$	W
	$V_2(x_1, x_2)_{\,x_2=-a_2} = 0$	L		$\varphi_2(x_1, x_2)_{\,x_2=-a_2} = 0$	V
SSCC	$w(x_1, x_2)_{\,x_1=-a_1} = 0$	W	CCCF	$w(x_1, x_2)_{\,x_1=-a_1} = 0$	W
	$M_{11}(x_1, x_2)_{\,x_1=-a_1} = 0$	X		$\varphi(x_1, x_2)_{\,x_1=-a_1} = 0$	U
	$w(x_1, x_2)_{\,x_2=a_2} = 0$	W		$w(x_1, x_2)_{\,x_2=a_2} = 0$	W
	$M_{22}(x_1, x_2)_{\,x_2=a_2} = 0$	Y		$\varphi_2(x_1, x_2)_{\,x_2=a_2} = 0$	V
	$w(x_1, x_2)_{\,x_1=a_1} = 0$	W		$w(x_1, x_2)_{\,x_1=a_1} = 0$	W
	$\varphi_1(x_1, x_2)_{\,x_1=a_1} = 0$	U		$\varphi_1(x_1, x_2)_{\,x_1=a_1} = 0$	U
	$w(x_1, x_2)_{\,x_2=-a_2} = 0$	W		$M_{22}(x_1, x_2)_{\,x_2=-a_2} = 0$	Y
	$\varphi_2(x_1, x_2)_{\,x_2=-a_2} = 0$	V		$V_2(x_1, x_2)_{\,x_2=-a_2} = 0$	L

Results. Similar to J. Reddy [13, 14], we will write notation boundary conditions at the edges of a rectangular plate using symbols CFSC, FFCC and so on.We begin definition of the boundary conditions from the left vertical clockwise. The next designations of the plate edge are introduced here: S – edge is simply supported, C – edge is clamped and F – edge is free. Among the solved examples, there are solutions of 6 of them: SSSC, SSSF, SSCC, SCSF, SCCC, CCCF. In the process of modeling of the structure, the following boundary conditions were imposed.

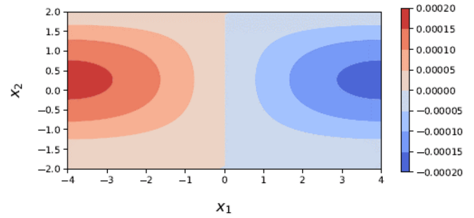
Following the geometric and mechanical parameters being taken in calculations: intensity of the load q_0 = 10 kPa, plate sizes $2a_1$ = 8 m, $2a_2$ = 4 m, thickness h = 0.2 m. Young's modulus is equal to $E = 30 \times 10^9$ Pa and Poisson ratio ν = 0.2.

3.1. Examples

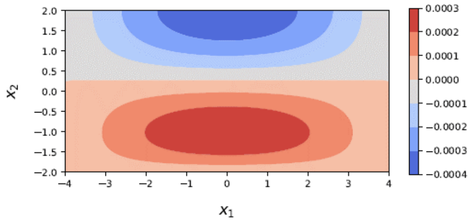
3.1.1. Plate SSSC



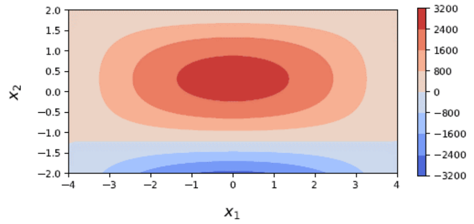
(a) Function w



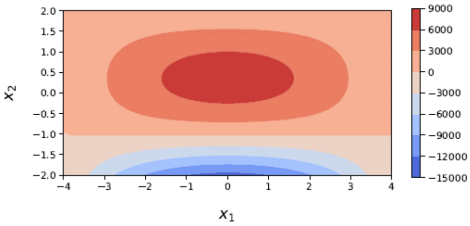
(b) Function φ_1



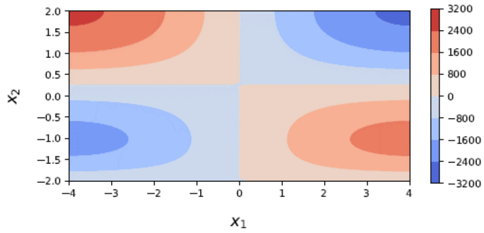
(c) Function φ_2



(d) Function M_{11}



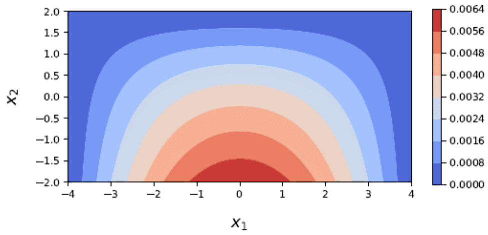
(e) Function M_{22}



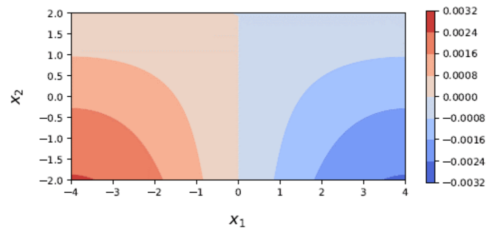
(f) Function M_{12}

Fig. 3. Results for a plate SSSC

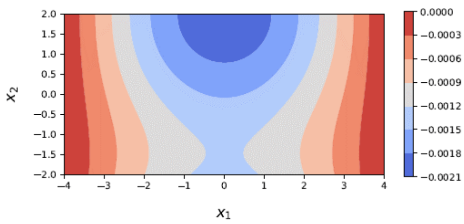
3.1.2. Plate SSSF



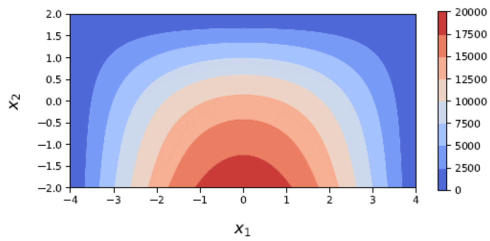
(a) Function w



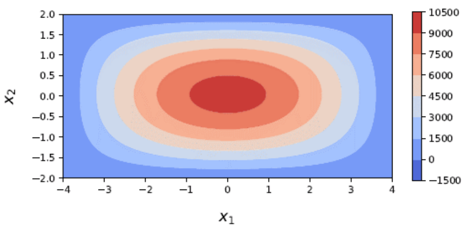
(b) Function φ_1



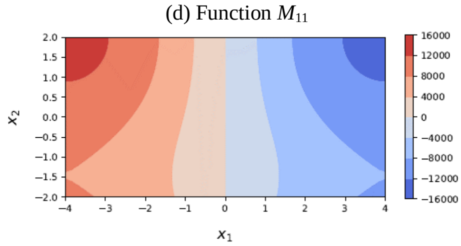
(c) Function φ_2



(d) Function M_{11}



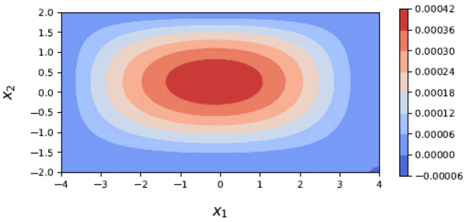
(e) Function M_{22}



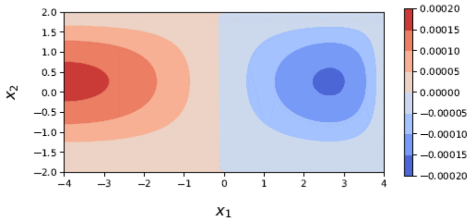
(f) Function M_{12}

Fig. 4. Results for a plate SSSF

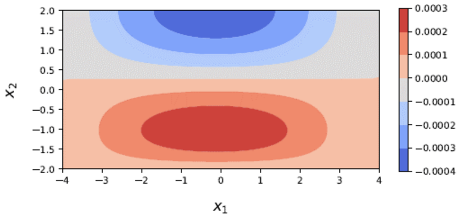
3.1.3. Plate SSCC



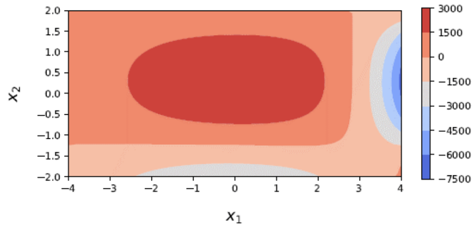
(a) Function w



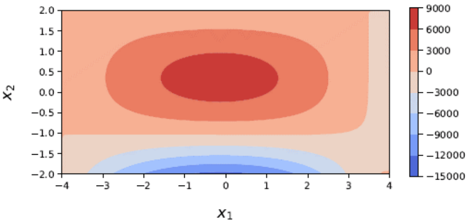
(b) Function φ_1



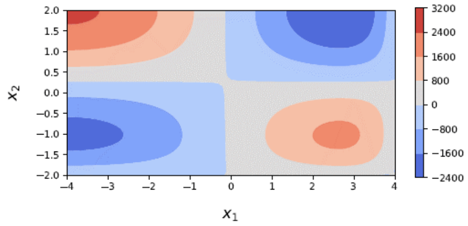
(c) Function φ_2



(d) Function M_{11}



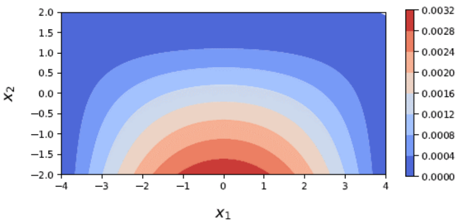
(e) Function M_{22}



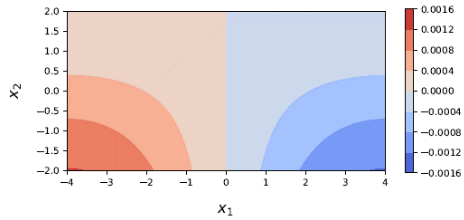
(f) Function M_{12}

Fig. 5. Results for a plate SSCC

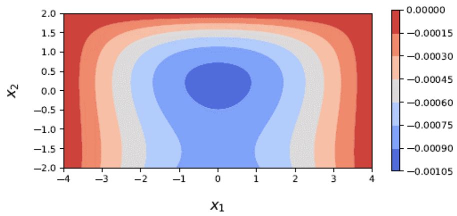
3.1.4. Plate SCSF



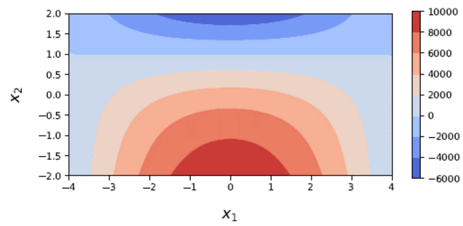
(a) Function w



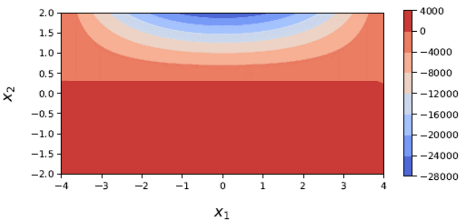
(b) Function φ_1



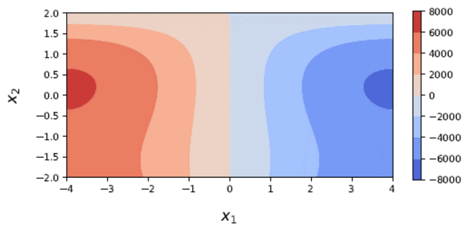
(c) Function φ_2



(d) Function M_{11}



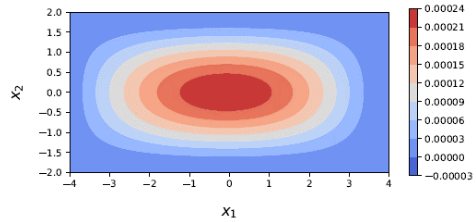
(e) Function M_{22}



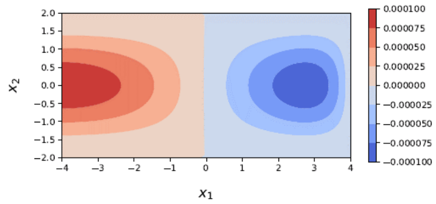
(f) Function M_{12}

Fig. 6. Results for a plate SCSF

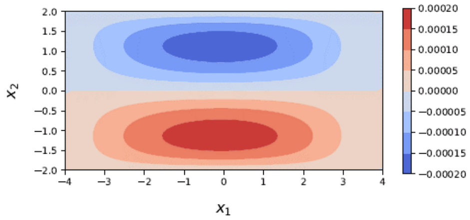
3.1.5. Plate SCCC



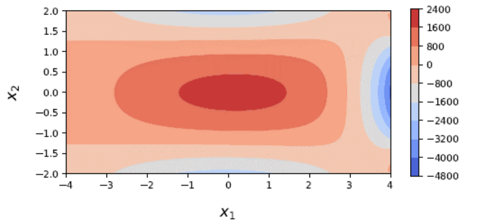
(a) Function w



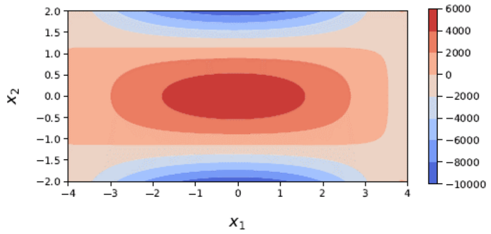
(b) Function φ_1



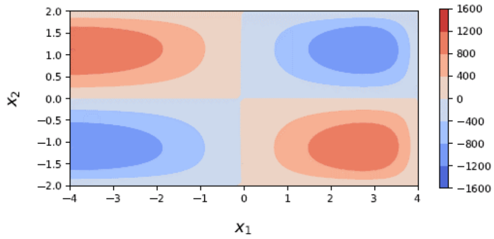
(c) Function φ_2



(d) Function M_{11}



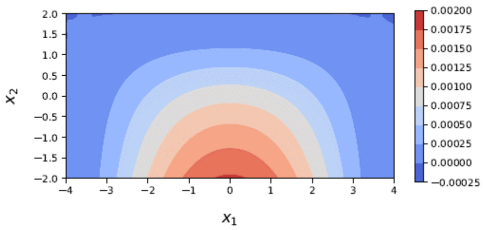
(e) Function M_{22}



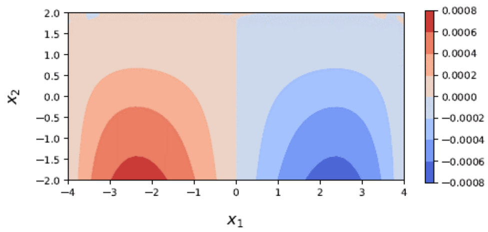
(f) Function M_{12}

Fig. 7. Results for a plate SCCC

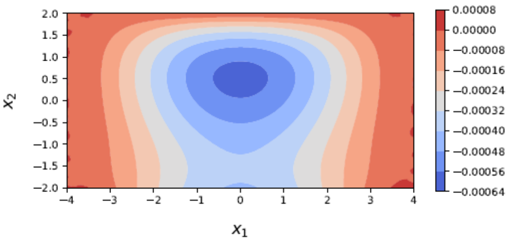
3.1.6. Plate CCCF



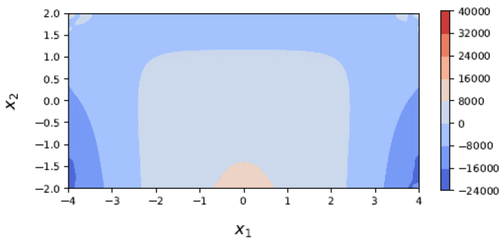
(a) Function w



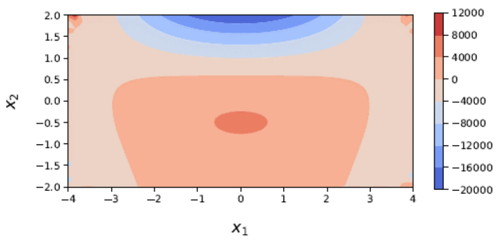
(b) Function φ_1



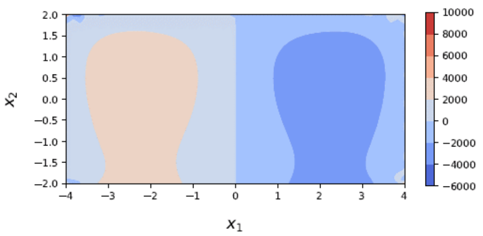
(c) Function φ_2



(d) Function M_{11}



(e) Function M_{22}



(f) Function M_{12}

Fig. 8. Results for a plate CCCF

Conclusions. The major findings of this research are summarized in the following:

1. The design model of thin isotropic plates under non-symmetrical boundary conditions is constructed. The most important elements of the model are: basic functions, functions of state of displacements and stresses in the plate, shape functions and force ones, edge node and surface ones, loading functions.

2. Within the model effective analytical-numerical method to solution of plate structure have been suggested. In contrary to FEM, the presented method is based on continuum model of material. For this reason, the operations like the structure discretization and finite element aggregation are unnecessary. It is characterized by simplicity of the structure modeling; possibility to define static, kinematic and mixed boundary conditions; high accuracy and efficiency of calculation; meshless approach for solving the problem.

The method allows:

• to obtain an exact general solution of equilibrium equation. A particular solution is obtained with high accuracy in the separate surface nodes,

• to generate set of the initial points at the coordinate axis and to distribute them at the plate contour,

• to write boundary conditions at each edge nodes. The number of them always corresponds to the number of unknown parameters of the model.

Displacements, slopes, moments and shearing forces are obtained by using the method of automatic differentiation. All operations are performed automatically with the author's program.

Effectiveness of the method is illustrated by the examples of rectangular plates with various non-symmetrical boundary conditions. The best results are obtained for an uniform distribution of edge nodes.

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